Introduction

Many elementary teachers would rank place value and regrouping as among the most important mathematics topics they teach each year. At the middle school level, the topic of place value is revisited as students work to understand decimals. And many teachers would add that these topics—place value, regrouping, and decimals—are among the most troublesome. Young students frequently lose track of the notion that the 4 in 45 stands for 4 tens, for instance. Even at the higher grades, students often have a hard time remembering correct procedures for calculating. Upper-grade teachers frequently report that although their students have been subtracting multidigit numbers for several years, some persist in subtracting the smaller from the larger, even if the larger digit is the one to be subtracted. For example, these students would claim that 342 minus 296 equals 154 instead of 46. And many middle school students ask, "Where do I place the decimal point when I'm adding, and where does it go when I'm multiplying?"

On the surface, it may seem mystifying that so many students, year after year, have such difficulty with these concepts. However, once we delve more deeply into an examination of the ideas underlying the base-ten number system, we see that those ideas are quite complex. When students are having difficulty learning the mathematics of the curriculum, the problem is not resolved by dismissing the students as "not made of the right stuff."

Indeed, the errors consistently made by many students do have elements of logic in them. For example, when children write ninety-five as 905, they are applying to written numerals what they understand about spoken numbers. By listening to such children and trying to understand how their ideas make sense, we come to see more clearly the differences between our spoken and written systems for representing number; we also identify the ideas that children need to work through.

What are the principles underlying our number system? How do children come to understand these principles? What are the connections that allow students to flexibly maneuver their way around the number system? A second-grade teacher, pondering these questions, wrote the following:

"Suppose that children can tell you that one place is called the ones place, one the tens place, and so on, and that if you have 10 ones, you can't write it in the ones column alone, but you need to write it in the tens column as well because 10 ones can be traded in for 1 ten and

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0 ones. (Jeez, that sounds ridiculously complex!) That knowledge, in my experience, does not ensure that they are able to break numbers apart into tens and ones to make sense of adding double-digit numbers."

This casebook for the Developing Mathematical Ideas (DMI) seminar *Building a System of Tens* is designed to help groups of teachers, as well as others engaged in elementary and middle school mathematics education, explore the structure of the base-ten number system and the ways students come to understand it. The cases have been written by elementary and middle school teachers describing episodes from their own classrooms. The teacher-authors, who are themselves working to understand the big ideas of the mathematics curriculum, have written these cases as part of their own process of inquiry. They come together on a regular basis to read and discuss one another's work.

The cases are grouped to present students in different classrooms who are working on similar mathematical issues related to the base-ten structure of number. Through the cases, we can study children's initial ideas as they talk about how numbers are written and decomposed; the students' application of those ideas as they figure out how to add, subtract, multiply, and divide; and the ways students extend those ideas as they think about decimal fractions.

In chapter 1, the cases look into five classrooms, second through sixth grades, to see how students think about adding and subtracting multidigit numbers. What is it that students must already understand in order to work with numbers in these ways? This question motivates the rest of the casebook. In chapter 2, the cases present students who are working on basic concepts regarding the place value structure of whole numbers. After some of those concepts have been identified, the cases in chapter 3 once again consider how students bring their understandings of base ten to addition and subtraction. How students apply these basic concepts and how these concepts can be extended to the work of upper-elementary and middle grades is the focus of chapters 4 through 7.

Chapter 8, the last in this casebook, is the essay "Highlights of Related Research." This essay summarizes some research findings that touch on the issues explored in the cases (chapters 1–7).

When this DMI seminar was first taught, many seminar participants reported that they had to learn how to read the cases:

> "It's different from reading a story. I feel as if I've had to comb through each episode with a fine-toothed comb."
>
> "I'm reading cases very slowly and I'm writing down thoughts about what I'm seeing in the text."

> "I find that when I do the same mathematics problems the students in the case are working on, I can better understand what they are doing."

Several teachers offered advice to other participants:

> "Read all the cases in a chapter once and try to write down the mathematical issues they raise. You might focus attention on two or three children that interest you and really figure out the mathematical issues that these few children are facing. Try to really understand how the children are thinking. Here's another way do it: After reading all the cases in a chapter, go over them again, looking for common threads. What mathematical issues connect these cases together?"

> "Remember that these are glimpses of *real kids* dealing with *real* situations, struggling to make sense of very difficult concepts. Pay particular attention to the *natural* ways students often solve problems."

> "Begin with the chapter introduction—it alerts you to the ideas you should pay attention to."

> "If possible, discuss the cases informally with other participants before the sessions. If you're unable to do any or some of the above, by all means, still come to the seminar!"

As the seminar proceeds, you might talk to other participants about the ways they read the cases to prepare for seminar discussions.