CHAPTER

7

Expanding ideas about division in the context of fractions

Case 28
Who says that's not the right equation? My own experience vs. students' thinking
Sarita
Grade 6, May

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Stretching elastic
Selena
Grade 6, November

The cases in chapter 7 present the work of two sixth-grade classes as they solve problems that their teachers identify as division of fractions. These cases not only provide us an opportunity to study division of fractions for ourselves but they also return us to the themes of this seminar:

- What does it mean to model a situation with a piece of arithmetic?
- As we see the various number sentences that could represent a single situation, what do we learn about the operations?
- How are our ideas about the operations extended as we encounter new situations they might model?

case 28

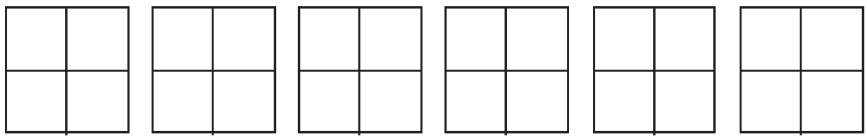
Who says that's not the right equation? My own experience vs. students' thinking

Sarita
GRADE 6, MAY

The students in my sixth-grade class have been solving fraction word problems for several weeks. Today they were spread around the classroom, working in groups of twos and threes. The directions were to draw a picture to solve the problem, write a number sentence (equation) that explains the problem, and show any computation. This was the problem I gave them to solve:

> You are giving a party. You have 6 pints of ice cream for the party. If you serve $\frac{3}{4}$ of a pint of ice cream to each guest, how many guests can be served?

For the most part, the students didn't find this a difficult problem. They drew pictures to represent the pints of ice cream and separated each pint into 4 equal sections. They explained that every 3 sections were what one person would receive, and that was equal to $\frac{3}{4}$. There was enough to do that 8 times, so 6 pints would serve 8 people.



However, when they expressed this problem in a number sentence or equation, the range of answers was interesting. I use the word interesting because I am not sure whether or not the students are incorrect or they just see things differently. Although I could understand the thinking behind their equations, I didn't think that they were "correct." I had been taught that equations have to match what is going on in the problem. In my mind the students' equations were, though not necessarily wrong, not quite correct, either.

I was trying to get the students to see that the problem could be solved by $6 \div \frac{3}{4}$. Instead, they came up with the following equations and had impressive reasoning to justify their thinking:

Equation	Justification
$24 \div 3 = 8$	There are 24 pieces, 3 pieces to a serving, 8 people can be served.
$8 \times \frac{3}{4} = 6$	8 servings of $\frac{3}{4}$ of a pint each gives you 6 whole pints.
$\frac{3}{4}+\frac{3}{4}+\frac{3}{4}+\frac{3}{4}+$ $\frac{3}{4}+\frac{3}{4}+\frac{3}{4}+\frac{3}{4}=6$	$\frac{3}{4}$ each gives you 6 whole pints.
$6-\frac{3}{4}-\frac{3}{4}-\frac{3}{4}-\frac{3}{4}-$ $\frac{3}{4}-\frac{3}{4}-\frac{3}{4}-\frac{3}{4}=0$	Take $\frac{3}{4}$ pint for each serving. You can do this 8 times.

Students also presented some other equations along with those listed above. Our routine is to leave all ideas out for discussion until the students are sure that a particular idea doesn't make sense. Thus, the following equations remained on the board even though I could not accept their logic. 25

Equation	Justification
$8 \div \frac{1}{4} = 6$	8 people were served, they had 6 pints, and the picture shows fourths.
$6 \div \frac{1}{4} = 8$	[*This was proposed in response to students who insisted that they couldn't accept the previous equation.*]
$24 \div \frac{3}{4} = 8$ or 6	There are 24 pieces altogether, and each serving is $\frac{3}{4}$ of a pint, so there are 6 pints or 8 servings (depending on what you are looking for).
$\frac{3}{4} \div 8 = 6$	$\frac{3}{4}$ pint is the serving, there are 6 pints of ice cream, so 8 servings.

These last few equations do not make sense mathematically, nor do the justifications help them make sense. In some ways these were the easy equations to dismiss. For the last one, I might be persuaded to help this student reexamine whole number division, because the justification seems clear—it just doesn't go with the equation.

My dilemma is that the first set of equations does not feel "right" to me even though they make mathematical sense. The 24 ÷ 3 = 8 is disturbing. I see what the students mean, and I understand what they are doing; but these are whole numbers, and I think this is a fraction problem. Can it be both?

Where do I go from here? What do I do with my own thinking? What do I do to get the students to see this as a fraction division problem? Or do I? Who is confused, the students or me?

case 29

Stretching elastic

Selena

GRADE 6, NOVEMBER

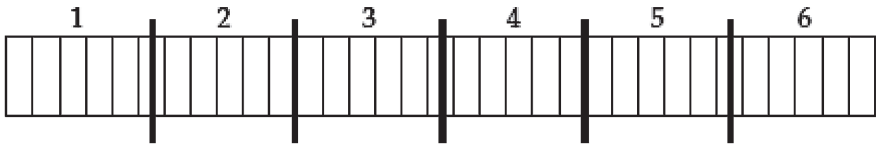
For a recent homework assignment, students were asked to solve a problem by drawing a diagram and writing an equation. When they arrived in mathematics class the next day, I placed them randomly in groups of two or three for a discussion of how everyone in their group solved the problem. Part of the homework asked for alternative solutions to the problem, so students knew that listening to other people's ideas would help them finish the worksheet. They also knew that I would then select a group of students to come to the board to explain how they solved the problem. This was the problem we were working on:

> A piece of elastic can be stretched to $5\frac{1}{2}$ times its original length. When fully stretched, it is 33 meters long. What was the elastic's original length?

The first group to come to the board solved the problem with the following diagrams, first showing the length of the stretched elastic:

Marcus: This is the 33 meters. We want to put them in $5\frac{1}{2}$ groups.

The group proceeded to separate the 33 meters of elastic into sections $5\frac{1}{2}$ meters long.



Before they could continue, hands all around the room flew into the air. Marcus called on Deena.

Deena: I agree with the 6, but I disagree with what Marcus said.

Teacher: What do you mean?

Deena: I know what Marcus is doing, but I think he is all mixed up.

Teacher: Can you say more about that?

I was not sure what Deena was thinking or what she was trying to articulate. It was difficult for her to express what she meant, but I wanted to give her the opportunity to explain her ideas.

Deena: OK. The 6 is the answer. I know that because there are 6 groups. Marcus said that they put the 33 meters into $5\frac{1}{2}$ groups, but they didn't do that.

Teacher: Can someone tell us in other words what Deena is saying?

Amir: I think Deena is saying that Marcus's group said one thing but did another. It doesn't really matter.

Patrice: But if it doesn't matter, where did the 6 come from anyway? You don't know about the 6 until you solve the problem.

Francine: [*She is Marcus's partner.*] We didn't know about the 6 until we finished with the groups. I think Marcus meant to say that we kept counting $5\frac{1}{2}$ pieces, or meters, until we ran out of meters. When we finished, we had 6 as the answer.

Teacher: Six what?

Marcus: Six meters. I see what they mean though. I didn't make $5\frac{1}{2}$ groups; I made 6 groups of $5\frac{1}{2}$.

Teacher: Can the problem be solved if we made $5\frac{1}{2}$ groups?

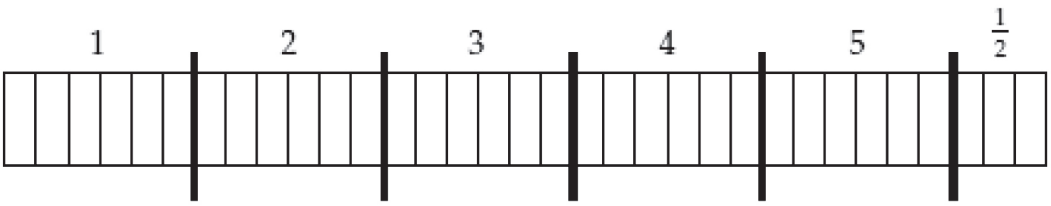
The students quickly discussed this question and began drawing new diagrams on their papers. Many were excited to think that I might have provided them with the alternative solution they were required to include on their homework papers.

I wondered if the students would be able to think about the commutative property of $5\frac{1}{2}$ groups of 6 compared with 6 groups of $5\frac{1}{2}$. I also wondered what would happen with the half when students tried to create $5\frac{1}{2}$ groups. How would they think about the half, and how would they express the half in their diagram?

When the noise settled to a calm hum, I called on another group.

Sonia: The answer will be the same no matter which way you do the problem. It doesn't matter because $6 \times 5\frac{1}{2}$ is the same as $5\frac{1}{2} \times 6$. Both of those are 33. I suppose you want a picture?

Sonia is one of my more proficient mathematics students. She tends to be quick to understand, quick to formulate an answer, and is usually not interested in drawing a diagram. However, I generally pretend I can accept nothing unless it can be proven with a diagram. So Sonia's group drew this on the board:



Victor: This picture shows $5\frac{1}{2}$ groups. There are 6 meters in each group except the last group. The last group only has 3 meters in it, because it is a half a group.

Teacher: What about an equation? [*Sonia wrote two equations on the board.*]

$$33 \div 5\frac{1}{2} = 6 \qquad 33 \div 6 = 5\frac{1}{2}$$

Daphne: We didn't do it that way, but I think it is the same thing. We kept adding.

$$5\frac{1}{2} + 5\frac{1}{2} + 5\frac{1}{2} + 5\frac{1}{2} + 5\frac{1}{2} + 5\frac{1}{2} = 33$$

Teacher: Is that the same, or is that different?

Daphne: It's a different way of doing the same thing.

Isaiah: You could subtract also. Just keep doing subtraction until you have zero. Oh! You would have to subtract six times!

$$33 - 5\frac{1}{2} - 5\frac{1}{2} - 5\frac{1}{2} - 5\frac{1}{2} - 5\frac{1}{2} - 5\frac{1}{2} = 0$$

The mathematics class was minutes from being over. Many of the students seemed to be thinking about the meaning of the operations of division and multiplication. I knew that there were many things I would like to have them discuss. Having a fraction in the problem did not seem to pose much of an obstacle; I wondered if that was because the fraction was a half, and many students had a special way of thinking about halves that did not always carry over to other fractions. In the days to come, I hope to test this theory.

But of course, the students could not let me rest on my laurels too long. In the last two minutes of class Philip confronted me with a new wrinkle.

Philip: When I do this the real way, I get a different answer: 6 remainder 3. [*He came to the board to show us.*] The 3 is half of 6. So the remainder is the half.

$$\begin{array}{r} 6r3 \\ 5\frac{1}{2}\overline{)\,33} \\ \underline{30} \\ 3 \end{array}$$

As everyone left, I thought to myself, "Half of what?" Now what do I do? Is this my lesson for tomorrow?